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M-Point Boundary Value Problem for Caputo Fractional Differential Eqautions

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***Abstract:* -** Sufficient conditions for the existence of solutions for a class of m-point boundary value problem involving Caputo fractional derivative are established using fixed point theorems. Banach fixed point theorem, Schaufer's fixed point theo- rem and Leray-Schauder type nonlinear alternatve are applied to study existence results.

Keywords: *- Boundary value problem, existence results, Caputo fractional derivative, fixed point theorems*

I. INTRODUCTION

Recently, many researchers are attracted towards fractional differential equations as many phenomena in various branches of science and engineering are modeled. Numer- ous applications are found in control systems, visco-elasticity, electrochemistry, phar- macokinetics, food science, etc [1, 2, 3, 20]. Significant contributions by researchers has been recorded in the monograph due to Kilbas et al [6]. Some results on the theory of fractional differential equations due to Lakshmikantham et. al. can be seen in [7, 8, 9, 10]. Periodic boundary value problem, integral boundary value problem and initial value problem for fractional differential equations of order q, $0 < q < 1$ was studied respectively by Ramirez and Vatsala [21], Wang and Xie [22] and Zhang [23]. Author developed monotone method for system of fractional differential equa- tions with various type of conditions involving Riemann-Liouville fractional derivative and Caputo fractional derivative of order q, $0 < q < 1$ and obtained existence and uniqueness results. [4, 11, 12, 13, 14, 15, 18, 19]. Benchora [2] in the year 2009 ob- tained sufficient conditions for the boundary value problem. Reently, author obtained sufficient conditions for the existence of solution of boundary value problems using fixed point theorems [16, 17]. In this paper sufficient conditions for the existence of solutions of the following m-point boundary value problem (BVP) involving Caputo fractional derivative are established via fixed point theorems.

$$^{c}D^{q}u(t) = f(t, u(t)) \quad \text{on} \quad J = [0, T] \qquad (1)$$

with the boundary conditions

$$u(0) = u_0, \quad u^{0}(0) = u_1 \quad u^{00}(0) = u_2, \quad u^{000}(0) = u_3, \quad \ldots \quad , u^{(m)}(T) = u_T \qquad (1)$$

where $^{c}D^{q}$ is the Caputo fractional derivative, $f : J \times R \to R$ is a continuous function and $u_0, u_1, u_2, u_3, ..., u_T$ are real constants.

II. PRELIMINARIES

Notation, definitions and preliminary results required in the later section are discussed here.
$C(J, R)$ denotes Banach space of all continuous functions from J into R with the norm
$\|u\|_{\infty} := sup\{|u(t)| : t \in J\}$.

Definition 2.1 *[3, 6] The fractional integral of a function $u(t)$ of order q is denoted by $I^q u(t)$. It is defined as*

$$I^{q}u(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}u(s)ds$$

where $\Gamma(.)$ is the Euler-Gamma function and $u \in L^1([a, b], R)$.

Definition 2.2 *[3, 6] The Caputo fractional derivative of $u(t)$ of order q is denoted by $^{c}D^{q}u(t)$. It is defined as*

$$ {}^{c}D^{q}u(t) = \frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1}u^{(m)}(s)ds, \quad m-1 \le q \le m, \quad m \in Z^{+} $$

Definition 2.3 *A function $u(t) \in C^m (J, R)$ with its q-derivative existing on J is said to be a solution of the problem if $u(t)$ satisfies the equation*

$$ {}^{c}D^{q}u(t) = f(t,u(t)) \quad on \quad J = [0,T] $$

and the initial conditions

$$ u(0) = u_0, \quad u'(0) = u_1, \quad u''(o) = u_2, \quad u^{iii}(0) = u_3, \ldots, u^{(m)}(T) = u_T $$

Following Lemmas play important role in the existence of solutions for the BVP (1)-(2).

Lemma 2.1 *[2] Let $q > 0$, then the fractional differential equation*

$$ {}^{c}D^{q}u(t) = 0 $$

has solution $u(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \ldots + c_n t^n = \sum_{i=0}^{n} c_i t^i$

for some $c_i \in R$, $i = 0, 1, 2, \ldots, n$, $n = [q]+1$.

Lemma 2.2 *[2] Let $q > 0$, then*

$$ I^{q}.{}^{c}D^{q}h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_n t^n $$

for some $c_i \in R$, $i = 0, 1, 2, 3, \ldots, n$, $n = [q]+1$.

III. EXISTENCE RESULTS

Existence result of the BVP (1)-(2) which is an immediate consequence of Lemma 2.1 and Lemma 2.2.

Lemma 3.1 *Let $m - 1 < q \le m$ and let $u(t) : J \to R$ be continuous. A function $u(t)$ is a solution of the fractional integral equation*

$$ u(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}u(s)ds - \frac{t^m}{m!\,\Gamma(q-m)}\int_0^T (T-s)^{q-m+1}u(s)ds + u_0 + u_1 t + \frac{u_2}{2!}t^2 + \ldots + \frac{u_m}{m!}t^m $$

(3)

if and only if $u(t)$ is a solution of the fractional BVP

$$ {}^{c}D^{q}u(t) = h(t) \quad t \in J \tag{4} $$

$$ u(0) = u_0, \quad u'(0) = u_1 \quad u''(0) = u_2, \quad u'''(0) = u_3, \ldots, \quad u^{(m)}(T) = u_T \tag{5} $$

Proof: Assume that $u(t)$ satisfies (4). Applying Lemma 2.1, we obtain

$$ u(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_m t^m + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds $$

$$ u'(t) = c_1 + 2c_2 t + 3c_3 t^3 + \ldots + mc_m t^{(m-1)} + \frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}h(s)ds $$

$$ u'(t) = 2c_2 t + 6c_3 t + 4.3c_4 t^2 + \ldots + m(m-1)c_m t^{(m-2)} + \frac{1}{\Gamma(q-2)}\int_0^t (t-s)^{q-3}h(s)ds $$

$$ u'''(t) = 6c_3 + 4.3.2c_4 t + \frac{1}{\Gamma(q-3)}\int_0^t (t-s)^{q-4}h(s)ds, \ldots, $$

$$ u^{(m)}(t) = m!\,c_m \frac{1}{\Gamma(q-m)}\int_0^t (t-s)^{q-m+1}h(s)ds $$

Using initial conditions, we get

$$c_0 = u_0, \qquad c_1 = u_1$$

$$c_2 = \frac{u_2}{2}, \qquad c_3 = \frac{u_3}{3!}, \ldots,$$

$$c_m = \frac{u_T}{m!} - \frac{1}{m!\Gamma(q-m)}\int_0^T (T-s)^{q-m+1} h(s)ds.$$

Hence

$$u(t) = u_0 + u_1 t + \frac{u_2}{2!}t^2 + \ldots + \frac{u_m}{m!}t^m - \frac{1}{m!\Gamma(q-m)}\int_0^T (T-s)^{q-m+1} h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} h(s)ds.$$

Conversely, assume that $u(t)$ satisfies fractional integral equation (3), then by definition of Caputo derivative, it follows that equation (4) and equation (5) also holds.

4. MAIN RESULTS

In this section we obtain results based on Banach fixed point theorem and Schaufer's fixed point theorem. Following result is obtained by using Banach fixed point theorem.

Theorem 4.1 *Assume that there exists a constant $k > 0$ such that*

$$\left| f(t,y) - f(t,\bar{y}) \right| \le k\left| y - \bar{y} \right|$$

for each $t \in J$ and all $y, \bar{y} \in \mathbb{R}$. If

$$kT^q\left[\frac{1}{\Gamma(q+1)} + \frac{1}{4!\Gamma(q-4)}\right] < 1, \tag{6}$$

then BVP (1.1)-(1.2) has a unique solution on J.

Proof: Transform the problem (1)-(2) into a fixed point problem. Define the operator $F : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by

$$F(u)(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,u)ds - \frac{1}{m!\Gamma(q-m)}\int_0^T (T-s)^{q-m+1} f(s,u)ds + u_0 +$$

$$+ u_1 t + \frac{u_2}{2!}t^2 + \frac{u_3}{3!}t^3 + \ldots + \frac{u_T}{m!}t^m$$

Clearly, the fixed points of the operator F are solutions of the problem (1)-(2). We shall use the Banach contraction principle to prove that F has a fixed point. Now, we shall show that F is a contraction mapping. Let $u, v \in C(J,\mathbb{R})$. Then for each $t \in J$, we have

$$\left| F(u)(t) - F(v)(t) \right| \le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left| f(s,u(s)) - f(s,v(s)) \right| ds +$$

$$+ \frac{1}{m!\Gamma(q-m)}\int_0^T (T-s)^{q-m+1}\left| f(s,u) - f(s,v) \right| ds$$

$$\le \frac{k\|u-v\|_\infty}{\Gamma(q)}\int_0^t (t-s)^{q-5} ds + \frac{k\|u-v\|_\infty T^4}{\Gamma(q-m)}\int_0^T (T-s)^{q-m+1} ds$$

$$\le \left[\frac{kT^q}{q\Gamma(q)} + \frac{k\Gamma^q}{m!\Gamma(q-m)}\right]\|u-v\|_\infty$$

$$= kT^q\left[\frac{1}{q\Gamma(q+1)} + \frac{1}{m!\Gamma(q-m)}\right]\|u-v\|_\infty$$

Thus

$$\| F(u)(t) - F(v)(t) \|_{\infty} \le kT^{q}\left[\frac{1}{\Gamma(q+1)} + \frac{1}{m!\,\Gamma(q-m)}\right]\| u - v \|_{\infty}$$

Consequently, by equation (6), F is a contraction. By Banach fixed point theorem, we claim that F has a fixed point which is a solution of the boundary value problem (1)-(2).
Following result is based on Schaefer's fixed point theorem:

Theorem 4.2 *Assume that*
(i) $f : J \times \mathrm{R} \to \mathrm{R}$ *is continuous*
(ii) *There exists a constant* $M > 0$ *such that* $|f(t,u)| \le M$ *for each* $t \in J$ *and all* $u \in \mathrm{R}$.
Then the BVP (1)-(2) *has at least one solution on* J.
Proof: We shall use Schaufer's fixed point theorem to prove that F has a fixed point. Now we prove:
(a)F is continuous:
Let u_n be a sequence such that $u_n \to u$ in $C(J,\mathrm{R})$. Then for each $t \in J$

$$| F(u_n)(t) - F(u)(t) | \le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} | f(s,u_n(s)) - f(s,u(s)) | ds +$$

$$+ \frac{1}{m!\,\Gamma(q-m)}\int_0^T (T-s)^{q-m+1} | f(s,u_n) - f(s,u) | ds$$

Since f is continuous function, we have

$$\|F(u_n - F(u)\|_{\infty} \to 0 \text{ as } n \to \infty.$$

(b)F maps the bounded sets into the bounded sets in $C(J,\mathrm{R})$:
It is enough to show that for any $\eta > 0$ there exists positive constant l such that for each $u \in B_{\eta} = \{u \in C(J,\mathrm{R} : \|u\|_{\infty} \le \eta\}$ we have $\|F(u)\|_{\infty} \le l$. By assumption (ii), we have for each $t \in J$

$$| F(u)(t) | \le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} | f(s,u) | ds - \frac{1}{m!\,\Gamma(q-m)}\int_0^T (T-s)^{q-m+1} | f(s,u) | ds$$

$$+ | u_0 | + | u_1 | t + | \frac{u_2}{2!} | t^2 + | \frac{u_3}{3!} | t^3 + \ldots + | \frac{u_m}{m!} | t^m$$

$$\le \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1} ds - \frac{T^m M}{m!\,\Gamma(q-m)}\int_0^T (T-s)^{q-m+1} ds + | u_0 | + | u_1 | T +$$

$$| \frac{u_2}{2!} | T^2 + | \frac{u_3}{3!} | T^3 + \ldots + | \frac{u_m}{m!} | T^m$$

$$\le \frac{MT^4}{\Gamma(q+1)} + \frac{MT^q}{\Gamma(q-m)} | u_0 | + | u_1 | T + | \frac{u_2}{2!} | T^2 + \ldots + | \frac{u_T}{m!} | T^m = l$$

Thus

$$| F(u)(t) | \le \frac{MT^q}{\Gamma(q+1)} + \frac{MT^q}{\Gamma(q-m)} + | u_0 | + | u_1 | T + | \frac{u_2}{2!} | T^2 + \frac{u_3}{3!} | T^3 | + \ldots + | \frac{u_m}{m!} | T^m : l$$

(c)F maps bounded sets into the equicontinuous sets $C(J,\mathrm{R})$:
Let $t_1, t_2 \in J$, $t_1 < t_2$, B_{η} be bounded set of $C(J,\mathrm{R})$ as in (b) and let $u \in B_{\eta}$. Then

$$| F(u)(t_2) - F(u)(t_1) | = \left| \frac{1}{\Gamma(q)}\int_0^{t_1} \left[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}\right] | f(s,u(s)) | ds + \right.$$

$$\left. + \int_{t_1}^{t_2} (t_2 - s)^{q-1} f(s,u(s)ds + u_1(t_2 - t_1) + \frac{u_2}{2!}(t_2^2 - t_1^2) + \frac{u_3}{3!}(t_2^3 - t_1^3) + \ldots + \frac{u_T}{m!}(t_2^m - t_1^m) \right|$$

$$\leq \frac{1}{\Gamma(q)}\int_0^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]|\ f(s,u(s))\ |\ ds\ +\frac{t_2-t_1}{2\Gamma(q-2)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\,|f(s,u(s))\ |\ ds\ +$$

$$+\ |\ u_1\ |\ (t_2-t_1)+\frac{|\ u_2\ |}{2!}(t_2^2-t_1^2)+\frac{|\ u_3\ |}{3!}(t_2^3-t_1^3)+\ldots+\frac{|\ u_T\ |}{m!}(t_2^m-t_1^m)$$

$$\leq \frac{M}{\Gamma(q+1)}\,|\ (t_2-t_1))^q+t_1^q-t_2^q\ |\ ds\ +\frac{M}{2\Gamma(q-1)}(t_2-t_1)^q\ \mathrm{s}+$$

$$+\ |\ u_1\ |\ (t_2-t_1)+\frac{|\ u_2\ |}{2!}(t_2-t_1)^2+\frac{|\ u_3\ |}{3!}(t_2-t_1)^3+\ldots+\frac{|\ u_T\ |}{m!}(t_2^m-t_1^m)$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. Using Arzela-Ascoli theroem, we conclude that $F : C(J,\mathrm{R}) \to C(J,\mathrm{R})$ is completely continuous.

(d) A priori bounds:

Now we show that $E=\{u\in C(J,R) : u=\lambda F(u)\ forsome\quad 0<\lambda<1\}$ is bounded.

Let $u \in E$, then $u = \lambda F(u)$ for some $0 < \lambda < 1$. Thus for each $t \in J$ we have

$$u(t)=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,u(s)ds-\frac{\lambda t^m}{m!\,\Gamma(q-m)}\int_0^T(T-s)^{q-m+1}f(s,u(s)ds\ +$$

$$+\lambda u_0+\lambda u_1 t+\lambda\frac{u_2}{2!}t^2+\lambda\frac{u_3}{3!}t^3+\ldots+\lambda\frac{u_T}{m!}t^m$$

This implies by assumption (ii) that for each $t \in J$ we have

$$|\ u(t)\ |\leq \frac{M}{q\Gamma(q)}T^4+\frac{M}{(q-m)\Gamma(q-m)}T^m+|\ u_0\ |+|\ u_1\ |\ T+\frac{u_2}{2!}T^2+\frac{u_3}{3!}t^3+\ldots+\frac{u_T}{m!}T^m$$

Thus for every $t \in J$ we have

$$\|u\|_\infty \leq \frac{M}{\Gamma(q+1)}T^q+\frac{M}{\Gamma(q+3)}T^q+|u_0|+|u_1|T+\frac{u_2}{2!}T^2+\frac{u_t}{3!}T^3+\ldots+\frac{u_T}{m!}T^{m\,4} := R.$$

This shows that E is bounded. As a consequence of Schaefer's fixed point theorem, we conclude that F has a fixed point which is a solution of the boundary value problem (1)- (2).

Following existence result for the BVP (1)-(2) is obtained by using Leray-Schauder type nonlinear alternatve.

Theorem 4.3 *Assume that*

(i) $f : J \times \mathrm{R} \to \mathrm{R}$ *is continuous*

(ii) *There exist* $\varphi_f \in L^1(J, \mathrm{R}^+)$ *and continuous and nondecreasing* $\psi : [0,\infty) \to (0,\infty)$ *such that* $|f(t,u)| \leq \varphi_f(t)\psi(|u|)$ *for each* $t \in J$ *and all* $u \in \mathrm{R}$.

(iii) *There exists a constant* $M > 0$ *such that*

$$\frac{M}{\left\|I^q\phi_f\right\|L_1 V(M)+P+|\ u_0\ |+|\ u_1\ |\ T+\frac{u_2}{2}T^2+\frac{u_T}{3}T^3+\ldots+\frac{u_T}{m}T^m}>1 \qquad (7)$$

$$where\ P = \frac{T^3}{3}(I^{q-3}\phi_f)(T)\psi(M) > 1$$

Then the BVP (1)-(2) *has at least one solution on* J.

Proof: Define the operator F as in Theorems 4.1 and 4.2. It can be shown that F is continuous and completely continuous. For $\lambda \in [0, 1]$, let u be such that for each $t \in J$

$$|u(t)| \leq \psi(\|u\|_\infty)\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\phi_f ds + \frac{\lambda T^m}{m!\,\Gamma(q-m)}\int_0^T (T-s)^{q-m+1}\phi_f ds +$$

$$+\lambda|u_0| + |u_1|T + \frac{|u_2|}{2!}T^2 + \frac{|u_3|}{3!}T^3 + \ldots + \frac{|u_T|}{m!}T^m$$

Thus

$$\frac{\|u\|\infty}{\psi(\|u\|_\infty)\|I^q\phi_f\|L^1 + \Delta + |u_0| + |u_1|T + \frac{|u_2|}{2}T^2 + \frac{|u_3|}{3}T^3 + \frac{|u_T|}{4}T^4} \leq 1,$$

where $\Delta = \frac{T^m}{m}(I^{q-3}\phi_f)(T)\psi(\|u\|\infty)$. Then by inequality (7), there exists M such that $\|u\|_\infty = M$.

Let $Y = \{u \in C(J,R) : \|u\|_\infty \leq M\}$.

The operator $F : \overline{Y} \to C(J,R)$ is continuous and completely continuous. By the choice of Y, there exists no $u \in \partial Y$ such that $u = \lambda F(u)$ for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [5], we deduce that F has a fixed point u in $\overline{Y}$, which is the solution of the BVP (1)-(2).

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